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Tensorial Representation of the Dirac Equation

MURRAY MURASKIN

University of North Dakota, Grand Forks, North Dakota

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Abstract

We discuss tensor representations of the Dirac equation using a geometric approach. We find that the mass zero Dirac equations can be represented by Maxwell equations having a source which obeys the empty space wave equation. We also obtain a relation for the source in terms of **E** and **H**. In the case of mass not equal to zero a difficulty is encountered in removing the constant spinors $\bar{\chi}_A$ and $\bar{\varphi}_A$. We find that the arbitrary constant spinors can be eliminated in a spinor theory based on the Klein-Gordon equation.

1. *Introduction*

Ruse (1937) has obtained a tensorial representation for the spinor Dirac equations by making use of a set of four linearly independent basis vector variables (geometric variables). However, his equations are, unfortunately, rather complicated, not exhibiting any simple structure. It is with the aim of improving on this work that we shall study spinor fields from a geometric viewpoint. In our geometric approach we find the relation between spinors and the Maxwell tensor is generalized from Klauder (1964) to involve the trace field (Muraskin, 1969). We find that the Maxwell equations with source satisfying the empty space wave equation (Muraskin, 1969) furnishes a tensorial representation of the mass zero Dirac equation. By requiring that the metric tensor, which is constructed from the spinor field in the manner of Ruse (1937), be the Minkowski metric, we obtain a relationship for the source of the Maxwell field (trace field) in terms of **E** and **H**.

When we study the mass m case (Whittaker, 1936; Klauder, 1964; Cercignani, 1967; Ruse, 1937), we encounter a difficulty in removing the constant spinors $\bar{\chi}_A$, $\bar{\varphi}_A$ from the theory. We find that the arbitrary constant spinors can be eliminated in a spinor theory based on the Klein-Gordon equation (Marx, 1967). In this situation the tensorial equations are themselves Klein-Gordon equations.

2. *Geometric Variables and Spinor Variables*

We define $\Lambda^A{}_B$ by

$$\begin{aligned} \chi_A &= \Lambda^B{}_A \bar{\chi}_B \\ \varphi_A &= \Lambda^B{}_A \bar{\varphi}_B \end{aligned} \tag{2.1}$$

with

$$\bar{\chi}_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\bar{\varphi}_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.2}$$

This choice of $\bar{\chi}_A$, $\bar{\varphi}_A$, although simple, is not meant to be unique. A unimodular transformation that corresponds to a constant Lorentz transformation can be performed on the spinor indices. Equations (2.1) and (2.2) give

$$\begin{aligned} \chi_1 &= \Lambda^1{}_1 \qquad \varphi_1 = \Lambda^2{}_1 \\ \chi_2 &= \Lambda^1{}_2 \qquad \varphi_2 = \Lambda^2{}_2 \end{aligned} \tag{2.3}$$

In (2.1) $\Lambda^A{}_B$ is interpreted as a transformation that takes χ_A, φ_A into the constant spinors $\bar{\chi}_A$, $\bar{\varphi}_A$. We construct from χ_A, φ_A the basis vector variables

$$\begin{aligned} l_i &= \sigma_i{}^{A\dot{B}} \chi_A \chi_{\dot{B}} \\ n_i &= \sigma_i{}^{A\dot{B}} \varphi_A \varphi_{\dot{B}} \\ m_i &= \sigma_i{}^{A\dot{B}} \chi_A \varphi_{\dot{B}} \\ \bar{m}_i &= \sigma_i{}^{A\dot{B}} \varphi_A \chi_{\dot{B}} \end{aligned} \tag{2.4}$$

We also take

$$\begin{aligned} e^1{}_i &= \frac{1}{\sqrt{2}}(-m_i - \bar{m}_i) \\ e^2{}_i &= \frac{1}{\sqrt{(2)}\, i}(\bar{m}_i - m_i) \\ e^3{}_i &= \frac{1}{\sqrt{2}}(n_i - l_i) \\ e^0{}_i &= \frac{1}{\sqrt{2}}(l_i + n_i) \end{aligned} \tag{2.5}$$

We take the spin metric to be

$$\epsilon_{AB} = \epsilon^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.6}$$

g_{ij} is defined in terms of the basis vector by

$$g_{ij} = e^\alpha{}_i e^\beta{}_j g_{\alpha\beta} = l_i n_j + n_i l_j - m_i \bar{m}_j - \bar{m}_i m_j \tag{2.7}$$

Since one normally associates the Dirac equation with Minkowski space, we require that the metric tensor (2.7) be given by the Minkowski metric

$$g_{ij} = (1, -1, -1, -1) \tag{2.8}$$

This implies, using (2.4) and (2.7), that $|\chi_A \varphi^A|^2 = 1$. The equation

$$\sigma^{\alpha A\dot{B}} = e^\alpha{}_l \sigma^{lC\dot{D}} \Lambda^A{}_C \Lambda^{\dot{B}}{}_D \tag{2.9}$$

is seen to be identically satisfied using (2.3), (2.4) and (2.5). Following Parke & Jehle (1965), we may solve (2.9) for $\Lambda^A{}_C$ in terms of $e^\alpha{}_i$. We get

$$\Lambda^A{}_C = f^\alpha{}_l \, \sigma_\alpha{}^{A\dot{D}} \, \sigma^l{}_{C\dot{D}} \exp(-i\alpha) \tag{2.10}$$

with

$$f^\alpha{}_l = \frac{e^\alpha{}_l}{(\mathrm{Tr}\, e^\alpha{}_l)^{1/2}} \tag{2.11}$$

where Tr stands for trace. Thus, from (2.1) we get

$$\begin{aligned} \chi_A &= \pm f^\alpha{}_l \, \sigma_\alpha{}^{B\dot{D}} \, \sigma^l{}_{AD} \, \bar{\chi}_B \exp(i\alpha) \\ \varphi_A &= \pm f^\alpha{}_l \, \sigma_\alpha{}^{B\dot{D}} \, \sigma^l{}_{AD} \, \varphi_B \exp(i\alpha) \end{aligned} \tag{2.12}$$

The $\pm$ is extracted from the right-hand side to explicitly show that χ_A, φ_A are not determined in sign from $e^\alpha{}_i$ [see equation (2.4)]. α, β indices are raised and lowered by the Minkowski metric $g_{\alpha\beta} = (1,-1,-1,-1)$ and ij indices are also raised and lowered by the Minkowski metric $g_{ij} = (1,-1,-1,-1)$. We shall require that an α, β Lorentz transformation induces the same Lorentz transformation on the i, j indices. Thus, there will be no distinction between the Greek and Latin indices, so we could write $f^\alpha{}_l \equiv f^\gamma{}_l$, as in Muraskin (1969). By expanding (2.12), we find that χ_A, φ_A are expressed in terms of $f_{[ij]} \equiv \frac{1}{2}(f_{ij} - f_{ji}), f = f^1{}_1 + f^2{}_2 + f^3{}_3 + f^0{}_0$ and α. Then, we have formally 8 $f_{[ij]}$, f and α expressed in terms of 8 χ_A and φ_A. α is itself not a geometric field, since it is not given in terms of $e^\alpha{}_i$. We can look at the elimination of $\exp(i\alpha)$ as a gauge transformation (see Appendix). We then get†

$$\begin{aligned} \chi_A &= \pm f^\alpha{}_l \, \sigma_\alpha{}^{B\dot{D}} \, \sigma^l{}_{A\dot{D}} \, \bar{\chi}_B \\ \varphi_A &= \pm f^\alpha{}_l \, \sigma_\alpha{}^{B\dot{D}} \, \sigma^l{}_{A\dot{D}} \, \bar{\varphi}_B \end{aligned} \tag{2.13}$$

Thus, in the geometric theory, the relation between spinors and the Maxwell tensor is a generalization of Klauder that involves the trace field.

3. *Tensorial Equations of $m = 0$ Dirac Equations*

We assume the mass zero Dirac equations for χ_A, φ_A

$$\begin{aligned} \sigma^{iAB} \, \partial_i \, \chi_A &= 0 \\ \sigma^{iAB} \, \partial_i \, \varphi_A &= 0 \end{aligned} \tag{3.1}$$

Inserting (2.13) into (3.1) gives the tensorial equation ($E_1 = f_{[10]}$, $E_2 = f_{[20]}$, $E_3 = f_{[30]}$, $H_1 = f_{[23]}$, $H_2 = f_{[31]}$, $H_3 = f_{[12]}$)

$$\begin{aligned} \nabla \cdot \mathbf{H} &= 0 \\ \nabla \cdot \mathbf{E} &= -\partial_0 \frac{f}{2} \\ \nabla \times \mathbf{E} &= -\partial_0 \mathbf{H} \\ \nabla \times \mathbf{H} &= \nabla \frac{f}{2} + \partial_0 \mathbf{E} \end{aligned} \tag{3.2}$$

† We assume that a gauge transformation is made when $e \neq 0$ and that the limit $e \to 0$ can then be taken.

Thus, we have obtained the Maxwell equations whose source obeys the empty space wave equation previously introduced in Section 1. We have obtained it from a geometric approach. From $|\chi_A \varphi^A|^2 = 1$ we get the condition

$$\frac{f^4}{16} + \frac{f^2}{2}(\mathbf{H}^2 - \mathbf{E}^2) + \mathbf{E}^4 + \mathbf{H}^4 - 2\mathbf{E}^2\mathbf{H}^2 + 4(\mathbf{E}.\mathbf{H})^2 = 1 \tag{3.3}$$

Thus, the source of the Maxwell field is a fixed function of **E** and **H**. The case of $f = 0$ is seen from (3.3) to be an additional restriction on **E**, **H**. The fact that f is real also, from (3.3), restricts the size of the **E**, **H** invariants.

If we attempt to impose the condition $\chi_A \varphi^A = 0$ which we have used previously (Muraskin, 1969), we find from (2.7), (2.6) and (2.4) that $g_{ij} = 0$. Thus, the condition $\chi_A \varphi^A = 0$ does not lead to an acceptable geometric theory. Although the plane wave spinors obeying $\chi_A \varphi^A = 0$ are useful in finding solutions of the Maxwell equations (2.8) having

$$\begin{aligned} \mathbf{E}.\mathbf{H} &= 0 \\ \frac{f^2}{4} &= \mathbf{E}^2 - \mathbf{H}^2 \end{aligned} \tag{3.4}$$

such spinors are not permitted in a geometric theory based on χ_A, φ_A.

4. *Summary of Mass Zero Problem*

We have thus obtained a tensorial representation of the mass zero Dirac equations by making use of a geometric approach. The tensor equations are independent of $\bar{\chi}_A$, $\bar{\varphi}_A$. They have a simple structure as contrasted with Ruse. The geometric approach leads to the source of the Maxwell equations being a particular function of **E** and **H**.

We now go on to study the mass m Dirac equation.

5. *Mass m Dirac Equation*

The Dirac equation in terms of two 2-component spinors can be written

$$\begin{aligned} \sqrt{(2)}\, \sigma^k{}_{A\dot{B}}\, \partial_k \varphi^{\dot{B}} &= -im\chi_A \\ \sqrt{(2)}\, \sigma^{kA\dot{B}}\, \partial_k \chi_A &= -im\varphi^{\dot{B}} \end{aligned} \tag{5.1}$$

The Dirac equation in terms of χ_A, φ_A takes the form

$$\begin{aligned} \sqrt{(2)}\, \sigma^{kA\dot{B}}\, \partial_k \chi_A &= -im\epsilon^{\dot{B}\dot{D}} \varphi_{\dot{D}} \\ \sqrt{(2)}\, \sigma^k{}_{A\dot{B}}\, \epsilon^{AD}\, \partial_k \varphi_D &= im\chi_{\dot{B}} \end{aligned} \tag{5.2}$$

From (5.2) we get

$$\begin{aligned} \Box\chi_A &= -m^2 \chi_A \\ \Box\varphi_A &= -m^2 \varphi_A \end{aligned} \tag{5.3}$$

The variables χ_A, φ_A are related to the geometric variables by (5.3). Inserting (2.13) into (5.2) we get a tensorial Dirac equation

$$\begin{aligned} \sqrt{(2)}\,\sigma^{kA\dot{B}}\,\sigma_\alpha{}^{C\dot{D}}\,\sigma^l{}_{A\dot{D}}\,\bar{\chi}_C\,\partial_k f^\alpha{}_l &= -im\epsilon^{\dot{B}\dot{D}} f^\alpha{}_l\,\sigma_\alpha{}^{E\dot{C}}\,\sigma^l{}_{E\dot{D}}\,\bar{\varphi}_{\dot{C}} \\ \sqrt{(2)}\,\sigma^k{}_{AB}\,\sigma_\alpha{}^{C\dot{D}}\,\sigma^l{}_{E\dot{D}}\,\epsilon^{AE}\,\bar{\varphi}_C\,\partial_k f^\alpha{}_l &= imf^\alpha{}_l\,\sigma_\alpha{}^{D\dot{C}}\,\sigma^l{}_{D\dot{B}}\,\bar{\chi}_{\dot{C}} \end{aligned} \tag{5.4}$$

The equations involve the arbitrary constant spinors $\bar{\chi}_A$, $\bar{\varphi}_A$. We shall evaluate the equation for

$$\bar{\chi}_A = \begin{pmatrix}1\\0\end{pmatrix} \qquad \text{and} \qquad \bar{\varphi}_A = \begin{pmatrix}0\\1\end{pmatrix}$$

We then get the following equations for **E** and **H**,

$$\begin{aligned} \nabla\cdot\mathbf{H} &= mE_3 \\ \nabla\cdot\mathbf{E} &= -\partial_0\frac{f}{2} + mH_3 \\ (\nabla\times\mathbf{E})_1 &= -\partial_0 H_1 + mH_2 \\ (\nabla\times\mathbf{E})_2 &= -\partial_0 H_2 - mH_1 \\ (\nabla\times\mathbf{E})_3 &= -\partial_0 H_3 - m\frac{f}{2} \\ (\nabla\times\mathbf{H})_1 &= \partial_1\frac{f}{2} + \partial_0 E_1 + mE_2 \\ (\nabla\times\mathbf{H})_2 &= \partial_2\frac{f}{2} + \partial_0 E_2 - mE_1 \\ (\nabla\times\mathbf{H})_3 &= \partial_3\frac{f}{2} + \partial_0 E_3 \end{aligned} \tag{5.5}$$

It can be checked from (5.5), or equivalently from (2.13) and (5.3), that

$$\begin{aligned} \Box f_{[ij]} &= -m^2 f_{[ij]} \\ \Box f &= -m^2 f \end{aligned} \tag{5.6}$$

In the limit of $m \to 0$ (5.5) goes into the equations obtained previously (3.2). The equations (5.5) are not covariant if we consider only transformations on the tensor indices of $f_{[ij]}$. This is because $\bar{\chi}_A$ and $\bar{\varphi}_A$ transform according to a unimodular transformation when the i, j indices are subject to a Lorentz transformation. In (5.5) $\bar{\chi}_A$ and $\bar{\varphi}_A$ do not explicitly appear, and thus the equation (5.5) only takes on a superficial appearance of noncovariance.

The equations (5.4) or (5.5) are rather clumsy, so one may ask whether a simple tensorial description can be obtained by our methods. We note that (5.6) is independent of $\bar{\chi}_A$ and $\bar{\varphi}_A$. A spinor theory based on the Klein-Gordon equation (5.3) has been discussed by Marx (1967). In such a theory, we see that the tensorial equations have a particularly simple form given by (5.6).

6. *Conclusions*

The Ruse formalism is a geometric formalism. That is, from χ_A, φ_A we can define the geometric variables $e^{\alpha}{}_i$, g_{ij}, Γ^i_{jk}. In the geometric formalism, we can gain further understanding of the quantity f introduced previously (Muraskin, 1969). We find that f is a prescribed function of **E**, **H**. The condition $f = 0$ is seen to be a restrictive condition on **E**, **H**.

We have found that the tensorial representation of the $m = 0$ Dirac equation and the tensorial representation of the Klein-Gordon spinor equation takes on simple forms. The mass m Dirac equation, we find, does not take on a simple form.

Appendix

Under a gauge transformation with gauge function α, we have

$$\psi = \begin{pmatrix} \chi_A \\ \varphi^{\dot{B}} \end{pmatrix} \to \psi \exp [i\alpha(x)]$$

The gauge transformation on ψ implies

$$\begin{aligned} \chi_A &\to \chi_A \exp [i\alpha(x)] \\ \varphi^A &\to \varphi^A \exp [-i\alpha(x)] \end{aligned} \tag{A.1}$$

l_i, n_i, m_i, $\bar{m}_i$ are unchanged if

$$\begin{aligned} \varphi_A &\to \varphi_A \exp [i\alpha(x)] \\ \chi_A &\to \chi_A \exp [i\alpha(x)] \end{aligned} \tag{A.2}$$

(A.1) and (A.2) are consistent if the spin metric behaves as follows, under a gauge transformation (since $\varphi^A = \epsilon^{AB} \varphi_B$)

$$\begin{aligned} \epsilon_{AB} &\to \epsilon_{AB} \exp [2i\alpha(x)] \\ \epsilon^{AB} &\to \epsilon^{AB} \exp [-2i\alpha(x)] \end{aligned} \tag{A.3}$$

The possibility for a phase function in the spin metric is mentioned by Parke & Jehle (1965). In addition, using $\chi^A = \epsilon^{AB} \chi_B$ we get the gauge properties of χ^A

$$\chi^A \to \chi^A \exp [-i\alpha(x)] \tag{A.4}$$

Thus, we have that the geometric variables $e^{\alpha}{}_i$ are gauge invariant. Also, the gauge transformation on χ_A, φ_A takes the form (A.2).

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